

Quantum Langevin equations and stability

Marc-Thierry Jaekel ^a and Serge Reynaud ^b

(a) *Laboratoire de Physique Théorique de l'ENS ^{*}, 24 rue Lhomond F75231 Paris Cedex 05 France*

(b) *Laboratoire de Spectroscopie Hertzienne de l'ENS [†], 4 place Jussieu F75252 Paris Cedex 05 France*
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Different quantum Langevin equations obtained by coupling a particle to a field are examined. Instabilities or violations of causality affect the motion of a point charge linearly coupled to the electromagnetic field. In contrast, coupling a scatterer with a reflection cut-off to radiation pressure leads to stable and causal motions. The radiative reaction force exerted on a scatterer, and hence its quasistatic mass, depend on the field state. Explicit expressions for a particle scattering a thermal field in a two dimensional space-time are given.

I. INTRODUCTION

Since its introduction for a description of Brownian motion [1], Langevin equation has been extended to a large variety of domains [2] and has led to many mathematical developments [3].

In a general way, Langevin equation describes the motion of a small system, with a few degrees of freedom (for instance its position \mathbf{q}), interacting with a bath composed of a very large number of degrees of freedom. Hence, the bath can be considered to exert a fluctuating force on the small system (of mass m). The force depends on the motion of the system, and for small displacements can be developed as the fluctuating force experienced by the system at rest (\mathbf{F}), plus a motional force proportional to the system's displacement:

$$m\ddot{\mathbf{q}}(t) = \mathbf{F}(t) + \int_{-\infty}^{\infty} dt' \chi(t-t') \mathbf{q}(t') \quad (1)$$

The fluctuating force (\mathbf{F}) is characterised by its time correlations, which are related to the motional susceptibility (χ) through a fluctuation- dissipation relation. For Brownian motion, the force fluctuations have white noise correlations, and are linked to the frictional force, which is proportional to the particle's velocity ($\dot{\mathbf{q}}$) [4,2]. Quantum versions have been developed, which preserve the main features of Langevin equation [5].

A charge coupled to a fluctuating electromagnetic field provides a natural example of a system obeying Langevin equation [6]. In this case, the frictional force is proportional to the third time derivative of the charge's position ($\ddot{\mathbf{q}}$). As is well known from classical electron theory, this reaction force is plagued with instabilities. The equations of motion possess 'runaway solutions', i.e. exponentially self accelerating motions. If specific boundary conditions are imposed to forbid such unstable solutions, pre-acceleration effects occur, the particle's motion anticipating on the applied force [7].

Even in vacuum and for neutral bodies, field fluctuations lead to macroscopic effects. Casimir forces and vacuum friction are such manifestations due to radiation pressure fluctuations [8,9]. Objects which scatter a quantum field in vacuum experience a frictional force when moving with non uniform acceleration [10]. A perfect reflector for a scalar field in a two dimensional (2d) space-time is submitted to a force proportional to the third time derivative of its position ($\ddot{\mathbf{q}}$). This force can be understood as the cumulative effect of radiation pressure fluctuations, and satisfies a fluctuation- dissipation relation [11].

The introduction of a frequency dependent scattering, with causality, unitarity and a high frequency transparency conditions, has provided a simple remedy for divergences induced by the vacuum fluctuations, of infinite energy [12]. This description also gives a treatment of Langevin equation related with vacuum radiation pressure, which is consistent and free from instabilities [13].

This approach is applied here to Langevin equations derived from radiation pressure fluctuations in a thermal state. In order to make the comparison with standard models of quantum Langevin equations explicit, a first part briefly recalls the properties related with fluctuation- dissipation relations and with instability, in the case of a linear coupling between particle and field. A second part describes the fluctuating radiation pressure and the radiative force exerted on a particle scattering a scalar field in a (2d) space-time. The dependence of the motional susceptibility on the field

^{*}Unité propre du CNRS associée à l'Ecole Normale Supérieure et à l'Université Paris-Sud

[†]Unité de l'Ecole Normale Supérieure et de l'Université Pierre et Marie Curie associée du CNRS

state is explicated in the case of thermal input states. Neglecting recoil effects in a consistent way is shown to lead to stable and causal motions.

II. LINEAR COUPLING

In this part, we recall some general properties of quantum Langevin equations resulting from linear coupling between a small system and a bath of oscillators. Emphasis will be put on the properties of the motional susceptibility χ and their consequences for the motions of the small system.

A. Linear response relations

Langevin equations are obtained by coupling a small system, with a few degrees of freedom, to a bath composed of a very large (infinite) number of degrees of freedom [5]. Eliminating the bath's variables in the equations of motion provides a reduced equation which involves, besides the small system's degrees of freedom, noise variables describing the initial (fluctuating) values of the bath's variables. In simple examples of quantum Langevin equations, the small system is linearly coupled to an infinity of harmonic oscillators representing the bath's degrees of freedom. A standard model is provided by a particle whose position q (or velocity) is linearly coupled to a scalar field in a (2d) space-time $\phi(t, x)$ (representing one polarisation of the electromagnetic field in a transmission line, for instance). For a harmonically bound non relativistic particle the Lagrangian can be written (units will be used such that light velocity is equal to 1; the non relativistic limit will be included in linear response $\dot{q} \ll 1$):

$$L = \frac{m}{2}\dot{q}^2 - \frac{K}{2}q^2 - \int_{-\infty}^{\infty} dx [a(x)\dot{q}\partial_t\phi + b(x)\dot{q}\phi + d(x)q\phi] + \int_{-\infty}^{\infty} dx \frac{1}{2}[\partial_t\phi^2 - \partial_x\phi^2] \quad (2)$$

(a dot meaning time differentiation, and noting that terms like $q\partial_t\phi$ are equivalent to terms like $\dot{q}\phi$). It will also be convenient to use Fourier transforms in space and time variables, which will be generally denoted:

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f[\omega] e^{-i\omega t}$$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} f[k] e^{ikx}$$

Lagrangian (2) leads to the following solution for the field:

$$\phi[\omega, k] = \phi^{in}[\omega, k] - \chi_{\phi\phi}[\omega, k] e[\omega, k] q[\omega]$$

with shorthand notation for coupling: $e[\omega, k] = \omega^2 a[k] - i\omega b[k] + d[k]$ (for real coupling $e[\omega, k]^* = e[-\omega, -k]$), where $\chi_{\phi\phi}$ is the retarded propagator of the field:

$$\chi_{\phi\phi}[\omega, k] = \frac{-1}{(\omega + i\epsilon)^2 - k^2} \quad (3)$$

and ϕ^{in} is a free input field. Using the canonical commutation relations, the retarded propagator can also be deduced from the free field commutator:

$$[\phi^{in}(t, x), \phi^{in}(t', x')] = 2\hbar \xi_{\phi\phi}(t - t', x - x') \quad (4)$$

$$\chi_{\phi\phi}(t, x) = 2i\theta(t) \xi_{\phi\phi}(t, x)$$

with:

$$\xi_{\phi\phi}[\omega, k] = \frac{\pi}{2k} [\delta(k - \omega) - \delta(k + \omega)]$$

$$= \text{Im}(\chi_{\phi\phi}[\omega, k]) \quad (5)$$

These identities relate the field susceptibility to an applied source (retarded propagator) with the field spectral density (field commutator) and are characteristic of linear response theory [14].

A quantum Langevin equation follows for the particle:

$$m\ddot{q}(t) - \int_{-\infty}^{\infty} dt' \chi(t-t')q(t') = -Kq(t) + F(t) \quad (6)$$

where χ (shortened notation for χ_{FF}) and F describe the motional susceptibility and the fluctuating force generated by coupling to the field:

$$\chi[\omega] = - \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e[\omega, k]e[-\omega, -k]}{(\omega + i\epsilon)^2 - k^2} \quad (7)$$

$$F[\omega] = - \int_{-\infty}^{\infty} \frac{dk}{2\pi} e[-\omega, -k]\phi^{in}[\omega, k]$$

The force commutator is deduced from the free field commutator (4):

$$\begin{aligned} [F(t), F(t')] &= 2\hbar\xi_{FF}(t-t') \\ \xi_{FF}[\omega] &= \frac{1}{4\omega}(e[\omega, \omega]e[-\omega, -\omega] + e[\omega, -\omega]e[-\omega, \omega]) \end{aligned} \quad (8)$$

Resulting from (5), a fluctuation-dissipation relation is also satisfied by the motional susceptibility and the force commutator (see (7) and (8)):

$$\xi_{FF}[\omega] = Im(\chi[\omega]) \quad (9)$$

The imaginary part of the susceptibility is related to the amount of dissipated energy in a stationary regime [15].

The motional force depends on the particle's position in a causal way (see (3) and (4)): $\chi(t)$ vanishes for negative values of t , or else, $\chi[\omega]$ is analytic in the upper half complex plane ($Im(\omega) > 0$). The causal properties of the motional susceptibility allow one to write a dispersion relation. When χ decreases sufficiently at infinity, this relation takes a simple form:

$$\begin{aligned} \chi[\omega] &= \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\xi_{FF}[\omega']}{\omega' - \omega - i\epsilon} \\ \chi(t) &= 2i\theta(t)\xi_{FF}(t) \end{aligned}$$

These properties can be used to determine the motional susceptibility from the force commutator. For linear coupling between the particle's position and the field, the motional susceptibility is determined by the field commutator, and does not depend on the field state. The classical and quantum Langevin equations then only differ by their noise [5].

Moreover, from the linear dependence of the force on the input field, there results that for a thermal input state the quantum noise is gaussian, like the classical one. The only difference then lies in the correlation function:

$$\langle F(t)F(t') \rangle - \langle F(t) \rangle^2 = C_{FF}(t-t') \quad (10)$$

(for a stationary input state). In quantum case, the force has a commutator satisfying a fluctuation-dissipation relation [14] (units are taken such that $k_B = 1$):

$$2\hbar\xi_{FF}[\omega] = C_{FF}[\omega] - C_{FF}[-\omega] = (1 - e^{-\frac{\hbar\omega}{T}})C_{FF}[\omega] \quad (11)$$

At zero temperature, only positive frequency modes contribute to the noise, as expected for the ground state:

$$C_{FF}[\omega] = 2\hbar\theta[\omega]\xi_{FF}[\omega]$$

In the limit of high temperature ($\hbar\omega \ll T$) the classical fluctuation- dissipation relation is recovered from (9) and (11):

$$Im(\chi[\omega]) = \frac{\omega}{2T}C_{FF}[\omega] \quad (12)$$

Usual Brownian motion corresponds to a force with white noise correlations (D is the momentum's diffusion coefficient):

$$C_{FF}[\omega] = 2D$$

There results a damping force proportional to velocity, with a friction coefficient related to the diffusion coefficient [16]:

$$\chi[\omega] = i\xi_{FF}[\omega] = \frac{D}{T}i\omega$$

B. Point charge

In three dimensional space, a similar situation to (2) is provided by a point charge e located at position¹ \mathbf{q} , harmonically bound and coupled to the electromagnetic potential $\mathbf{A}(t, \mathbf{x})$. In the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) and in the dipole approximation, the Lagrangian reads [17]:

$$L = \frac{m}{2}\dot{\mathbf{q}}^2 - \frac{K}{2}\mathbf{q}^2 - e\mathbf{A}(\mathbf{0})\dot{\mathbf{q}} + \frac{1}{8\pi} \int_{-\infty}^{\infty} d\mathbf{x} [\partial_t \mathbf{A}^2 - (\nabla \wedge \mathbf{A})^2]$$

Recoil effects are neglected, so that the system's position is considered to be linearly coupled to the electromagnetic potential, evaluated at the mean position of the charge ($\mathbf{A}(\mathbf{0}) = \mathbf{A}(t, \mathbf{0})$).

Eliminating the electromagnetic field in the resulting equations of motion, the point charge obeys a quantum Langevin equation similar to (6):

$$m\ddot{\mathbf{q}}(t) - \int_{-\infty}^{\infty} dt' \chi(t-t')\mathbf{q}(t') = -K\mathbf{q}(t) + \mathbf{F}(t) \quad (13)$$

with:

$$\begin{aligned} \chi[\omega] &= -\frac{4}{3}\omega^2 \int_0^{\infty} \frac{k^2 dk}{\pi} \frac{e^2}{(\omega + i\epsilon)^2 - k^2} \\ \mathbf{F}[\omega] &= e\dot{\mathbf{A}}^{in}(\mathbf{0})[\omega] \end{aligned}$$

where \mathbf{A}^{in} is a free input electromagnetic potential. The integral appearing in χ is divergent (the charge self-energy is infinite) and must be renormalised. A simple regulation is obtained by introducing a form factor $\Omega[k]$ which decouples the charge from the field modes whose frequency exceeds some large cut-off frequency Ω ([17,18]):

$$\begin{aligned} \chi[\omega] &= -\frac{4}{3}\omega^2 \int_0^{\infty} \frac{k^2 dk}{\pi} \frac{e^2 \Omega[k]}{(\omega + i\epsilon)^2 - k^2} \\ &= \frac{1}{2}\chi''[0]\omega^2 - \frac{4}{3}\omega^4 \int_0^{\infty} \frac{dk}{\pi} \frac{e^2 \Omega[k]}{(\omega + i\epsilon)^2 - k^2} \\ &= \frac{1}{2}\chi''[0]\omega^2 + \frac{2}{3}ie^2\omega^3 + O\left(\frac{1}{\Omega}\right) \end{aligned}$$

with:

$$\frac{1}{2}\chi''[0] = \frac{4}{3}e^2 \int_0^{\infty} \frac{dk}{\pi} \Omega[k]$$

A model of regulator is for instance:

$$\Omega[k] = \left(\frac{\Omega^2}{\Omega^2 + k^2}\right)^2$$

¹bold face letters denote vectors in three dimensional space

$$\chi[\omega] = -\frac{e^2}{3} \frac{\Omega^3 \omega^2}{(\omega + i\Omega)^2} \quad \frac{1}{2} \chi''[0] = \frac{e^2}{3} \Omega$$

The susceptibility tends to a constant at infinite frequency. Written in the frequency domain, the left-hand side of Langevin equation (13), behaves like $-m\omega^2 \mathbf{q}[\omega]$ at high frequencies, so that the bare mass m can be considered as a high frequency mass. At low frequencies, a further contribution comes from the field reaction, which induces a mass correction μ and leads to a different quasistatic mass M :

$$M = m + \mu \quad \mu = \frac{1}{2} \chi''[0] \quad (14)$$

In the infinite cut-off limit, the renormalised mass M of the charge remains finite while the induced mass μ becomes infinite, so that the bare mass must be infinitely negative [17,18]. This limit provides a Langevin equation where the motional force is the well-known radiative reaction force:

$$M \ddot{\mathbf{q}} - \frac{2}{3} e^2 \ddot{\ddot{\mathbf{q}}} = -K \mathbf{q} + \mathbf{F} \quad (15)$$

The left-hand side is the Abraham-Lorentz equation of classical electron theory [7].

Recalling the commutators of the free electromagnetic field ($k = |\mathbf{k}|$):

$$\begin{aligned} [\mathbf{A}_i^{in}(t, \mathbf{x}), \mathbf{A}_j^{in}(t', \mathbf{x}')] &= 2\hbar \xi_{\mathbf{A}_i \mathbf{A}_j}(t - t', \mathbf{x} - \mathbf{x}') \\ \xi_{\mathbf{A}_i \mathbf{A}_j}[\omega, \mathbf{k}] &= 4\pi(\delta_{ij} - \frac{\mathbf{k}_i \mathbf{k}_j}{k^2}) \xi[\omega, k] \end{aligned}$$

one can compute the susceptibility from the force fluctuations:

$$\begin{aligned} [\mathbf{F}_i(t), \mathbf{F}_j(t')] &= 2\hbar \delta_{ij} \xi_{\mathbf{FF}}(t - t') \\ \xi_{\mathbf{FF}}[\omega] &= \frac{4}{3} \omega^2 \int_0^\infty \frac{k^2 dk}{\pi} e^2 \Omega[k] \xi[\omega, k] \\ &= \frac{2}{3} e^2 \omega^3 \Omega[\omega] \end{aligned}$$

Using analyticity properties and the fluctuation-dissipation relation (9), one can recover the motional susceptibility from the electromagnetic field fluctuations. The regularised susceptibility tending to a positive constant at infinite frequency, the dispersion relation must be written with at least one subtraction [19]. The static susceptibility $\chi[0]$ vanishes by translation invariance, so that one can write:

$$\begin{aligned} \chi[\omega] &= \omega \int_{-\infty}^\infty \frac{d\omega'}{\pi} \frac{\xi_{\mathbf{FF}}[\omega']}{\omega'(\omega' - \omega - i\epsilon)} \\ &= -2\omega^2 \int_0^\infty \frac{dk}{\pi k} \frac{\xi_{\mathbf{FF}}[k]}{(\omega + i\epsilon)^2 - k^2} \end{aligned} \quad (16)$$

(as $\xi_{\mathbf{FF}}$ is an odd function of the frequency). The induced mass depends on the regulator and diverges in the infinite cut-off limit:

$$\mu = 2 \int_0^\infty \frac{dk}{\pi} \frac{\xi_{\mathbf{FF}}[k]}{k^3} \quad (17)$$

The bare mass must contain an (infinitely) negative counterterm (see 14), which leaves the quasistatic mass M undetermined.

C. Positivity and instability

The linear equations of motion for the small system (13) are easily solved in the frequency domain. Introducing the mechanical impedance Z and admittance Y of the system (the notation $f\{p\} = f[ip]$ relates Laplace with Fourier transforms), one obtains from (16):

$$Z\{p\} = mp + \frac{K}{p} - \frac{\chi\{p\}}{p} = Y\{p\}^{-1}$$

$$-\frac{\chi\{p\}}{p} = \int_0^\infty \frac{dk}{\pi} \frac{\xi_{\mathbf{FF}}[k]}{k} \frac{2p}{p^2 + k^2} \quad (18)$$

with: $\xi_{\mathbf{FF}}[k]/k \geq 0$.

The system's velocity is determined in terms of the applied force by:

$$\dot{\mathbf{q}}[\omega] = Y[\omega]\mathbf{F}[\omega] \quad (19)$$

According to causality, the motional force is a retarded function of the system's displacement. The susceptibility χ and the mechanical impedance Z , as functions of the frequency, are analytic in the upper half plane $Im(\omega) > 0$. No poles are present in the upper half plane, which could produce unbounded forces from a finite displacement of the system. Because of its 'closed loop gain' like expression, the admittance Y requires a closer examination. It is well known that the Abraham-Lorentz equation (15) possesses 'runaway solutions' leading to unstable motions [7].

The spectral decomposition (18) shows that the motional force for a point charge is related to the system's velocity through a positive function [20]. $-\chi/p$ is holomorphic in the complex half plane $Re(p) > 0$ and satisfies:

$$Re\left(-\frac{\chi\{p\}}{p}\right) > 0 \quad for \quad Re(p) > 0$$

According to (14, 17), the system's bare mass remains positive as long as the cut-off satisfies the inequality:

$$m \geq 0 \quad or \quad M \geq \mu \quad (20)$$

so that the system's impedance is also a positive function in this case. The inverse of a positive function is positive, and causality follows from positivity [20]. When inequality (20) is satisfied, the admittance is also a causal function, and no 'runaway solutions' can appear (see (19)). The quantum Langevin equation leads to stable motions in this case [17].

However, the renormalised impedance and admittance of the point charge are not positive functions, a consequence of the occurrence of a negative coefficient (m) in the spectral decompositions. Indeed, the renormalised expressions can be written:

$$Z[\omega] = -iM\omega + i\frac{K}{\omega} + \frac{2}{3}e^2\omega^2 = Y[\omega]^{-1}$$

showing that the admittance has a pole in the upper half plane at $\omega \sim i3M/2e^2$ (for $\hbar(K/M)^{\frac{1}{2}} \ll M$). The renormalised Langevin equation (15) leads to unstable self-accelerating motions of the system ('runaway solutions'). If specific boundary conditions are imposed to exclude these unphysical solutions, the system's motions can then be shown to anticipate on the applied force and to violate causality [7]. Positivity of the bare mass (20) is the condition for the Langevin equation to lead to stable and causal motions [17].

III. RADIATION PRESSURE

A neutral system which scatters a quantum field experiences a radiation pressure which vanishes in the average, but still fluctuates [21]. In particular, the radiation pressure is responsible for the Casimir forces between two bodies [22,12]. The fluctuating radiation pressure produces long term cumulative effects: a moving scatterer also experiences a mean force depending on its motion. Using quantum field theory, the motional force exerted on a mirror in the vacuum of a scalar field has been obtained [10]. For small motions, linear response theory [14] shows that the motional force is connected with the fluctuations of the radiation pressure at rest [11]. A point scatterer then obeys a Langevin equation of the form (1). We now study the case of a point system scattering a scalar field in a (2d) space-time.

A. Point scatterer

In two dimensional space-time, a point scatterer located at a position q separates space into two regions. In each of them the scalar field evolves freely and is the sum of two counterpropagating components:

$$\begin{aligned}\Phi(t, x) &= \phi_{in}(t - x) + \psi_{out}(t + x) & \text{for } x < q \\ \Phi(t, x) &= \phi_{out}(t - x) + \psi_{in}(t + x) & \text{for } x > q\end{aligned}$$

The outgoing fields are related to the incoming ones by a scattering matrix. Neglecting recoil effects, the S-matrix on the scatterer at rest will be written:

$$\begin{aligned}\phi_{out}[\omega] &= s[\omega]\phi_{in}[\omega] + e^{2i\omega q}r[\omega]\psi_{in}[\omega] \\ \psi_{out}[\omega] &= e^{-2i\omega q}r[\omega]\phi_{in}[\omega] + s[\omega]\psi_{in}[\omega]\end{aligned}\tag{21}$$

r and s are the reflection and transmission coefficients defined for the scatterer at rest at $q = 0$. Besides the reality, causality and unitarity properties of the S-matrix:

$$\begin{aligned}r^*[\omega] &= r[-\omega] & s^*[\omega] &= s[-\omega] \\ r, s & \text{ analytic for } \text{Im}(\omega) > 0 \\ |r|^2 + |s|^2 &= 1\end{aligned}$$

a transparency condition will be assumed, with a cut-off frequency corresponding to an energy smaller than the mass (M_0) of the scatterer:

$$\begin{aligned}r[\omega] &\sim 0 & \text{for } \omega \gg \omega_c \\ & & \hbar\omega_c \ll M_0\end{aligned}\tag{22}$$

This condition (satisfied by realistic mirrors) allows one to neglect recoil effects, and will play an important role in the following. A perfect reflector corresponds to $r = -1$ for all frequencies and does not obey the required conditions.

In each region, the energy (e) and momentum (p) densities of the field are those of a free scalar field:

$$\begin{aligned}e(t, x) &= \dot{\phi}^2(t - x) + \dot{\psi}^2(t + x) \\ p(t, x) &= \dot{\phi}^2(t - x) - \dot{\psi}^2(t + x)\end{aligned}$$

The radiation pressure exerted on the motionless scatterer is obtained from the stress tensor of the field, evaluated at the scatterer's position:

$$F(t) = \dot{\phi}_{in}^2(t - q) + \dot{\psi}_{out}^2(t + q) - \dot{\phi}_{out}^2(t - q) - \dot{\psi}_{in}^2(t + q)\tag{23}$$

and can be expressed in terms of the input fields and the S-matrix.

B. Radiation pressure fluctuations and radiative reaction

For input fields in a stationary and isotropic state, the field correlation functions can be written:

$$\begin{aligned}-\omega\omega' < \phi_{in}[\omega]\phi_{in}[\omega'] > &= -\omega\omega' < \psi_{in}[\omega]\psi_{in}[\omega'] > = 2\pi c[\omega]\delta(\omega + \omega') \\ < \phi_{in}[\omega]\psi_{in}[\omega'] > &= 0\end{aligned}$$

The field correlations can be decomposed into an antisymmetric part (free field commutator) which does not depend on the state and a symmetric part (mean field anticommutator) which is state dependent (see eq.(4); ξ is a shortened notation for $\xi_{\phi\phi}$):

$$\begin{aligned}-\omega\omega'[\phi_{in}[\omega], \phi_{in}[\omega']] &= -\omega\omega'[\psi_{in}[\omega], \psi_{in}[\omega']] = 4\pi\hbar\xi[\omega]\delta(\omega + \omega') \\ c[\omega] &= \hbar(\xi[\omega] + \sigma[\omega]) & \xi[\omega] &= \frac{\omega}{4}\end{aligned}$$

In particular, for a thermal input state the field correlations are given by a fluctuation-dissipation relation (see 11):

$$\begin{aligned}c[\omega] &= \frac{2\hbar\xi[\omega]}{1 - e^{-\frac{\hbar\omega}{T}}} = \frac{\hbar\omega}{2(1 - e^{-\frac{\hbar\omega}{T}})} \\ \sigma[\omega] &= \frac{\omega}{4} \coth \frac{\hbar\omega}{2T}\end{aligned}$$

The radiation pressure fluctuations are determined by the fluctuations of the input fields (see eqs (10) and (23)). In a thermal state, the mean quartic forms are obtained from the 2-point correlations using Wick's rules, and lead to (see eq.(25) of [11]; $\omega^2 c[\omega]$ has been changed to $c[\omega]$):

$$\begin{aligned} C_{FF}[\omega] &= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} 4c[\omega']c[\omega - \omega']\gamma[\omega', \omega - \omega'] \\ \gamma &= |\alpha|^2 + |\beta|^2 \\ \alpha[\omega, \omega'] &= 1 - s[\omega]s[\omega'] + r[\omega]r[\omega'] \\ \beta[\omega, \omega'] &= s[\omega]r[\omega'] - r[\omega]s[\omega'] \end{aligned} \quad (24)$$

The mean force commutator follows and satisfies fluctuation-dissipation relation (11). Noting that:

$$\gamma[\omega, \omega'] = \gamma[\omega', \omega] = \gamma[-\omega, -\omega']$$

$$(1 - e^{-\frac{\hbar\omega}{T}})c[\omega']c[\omega - \omega'] = \frac{\hbar^2}{2}\{\omega'\sigma[\omega - \omega'] + (\omega - \omega')\sigma[\omega']\}$$

it can also be written:

$$\xi_{FF}[\omega] = \hbar \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} 2(\omega - \omega')\sigma[\omega']\gamma[\omega', \omega - \omega']$$

This expression, which also results directly from (23) and the field commutator, exhibits the general dependence of the mean force commutator on the input state.

Motions of the point scatterer alter the field scattering. The S-matrix introduced in (21) describes the field scattering in the comoving frame and coordinate transformations must be used to recover the S-matrix in the original frame [11]. The expression of the force in terms of the scattered fields also suffers velocity dependent changes following the covariant nature of the field stress tensor. Considering only first order corrections in displacements, the radiation pressure exerted on a the moving scatterer is obtained under the form (1) with (see eq.(19) of [11]):

$$\chi[\omega] = 4i\hbar \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} (\omega - \omega')\sigma[\omega']\alpha[\omega', \omega - \omega']$$

For a stationary state, the static susceptibility vanishes (σ is an even function of ω):

$$\chi[0] = 0$$

Recalling the S-matrix unitarity, one remarks that $\gamma = 2Re(\alpha)$, so that fluctuation-dissipation (9) is satisfied by the radiation pressure fluctuations and the radiative reaction force.

In contrast to linear coupling, the radiative reaction force exerted on a point scatterer depends on the input field state. The susceptibility for a thermal state at temperature T :

$$\chi_T[\omega] = i\hbar \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \omega'(\omega - \omega') \left\{ 1 + \frac{2}{e^{\frac{\hbar\omega'}{T}} - 1} \right\} \alpha[\omega', \omega - \omega']$$

can be decomposed into a vacuum contribution and a thermal correction:

$$\begin{aligned} \chi_0[\omega] &= i\hbar \int_0^{\omega} \frac{d\omega'}{2\pi} \omega'(\omega - \omega') \alpha[\omega', \omega - \omega'] \\ \chi_T[\omega] &= \chi_0[\omega] + 2i\hbar \int_0^{\infty} \frac{d\omega'}{2\pi} \frac{\omega'}{e^{\frac{\hbar\omega'}{T}} - 1} \{ (\omega + \omega')\alpha[-\omega', \omega + \omega'] \\ &\quad + (\omega - \omega')\alpha[\omega', \omega - \omega'] \} \end{aligned} \quad (25)$$

The force responses to quasistatic motions (translation, constant velocity, constant acceleration, ...) are given by a Taylor expansion around zero frequency:

$$\chi'_0[0] = \chi''_0[0] = 0$$

The vacuum state is Lorentz invariant. Under uniformly accelerated motion its fluctuations appear as thermal ones in the comoving frame of the point scatterer [23]. Hence, the corresponding responses vanish. At non zero temperature, a friction coefficient related to viscosity and a correction to the quasistatic mass appear:

$$\begin{aligned}\chi'_T[0] &= 2i\hbar \int_0^\infty \frac{d\omega'}{2\pi} \frac{\omega'}{e^{\frac{\hbar\omega'}{T}} - 1} \{ (1 + \omega' \partial_{\omega'}) \alpha[\omega', -\omega'] \\ &\quad + (1 - \omega' \partial_{\omega'}) \alpha[-\omega', \omega'] \} \\ \frac{1}{2} \chi''_T[0] &= 2i\hbar \int_0^\infty \frac{d\omega'}{2\pi} \frac{\omega'}{e^{\frac{\hbar\omega'}{T}} - 1} \{ (1 + \frac{\omega'}{2} \partial_{\omega'}) \partial_{\omega'} \alpha[\omega', -\omega'] \\ &\quad + (1 - \frac{\omega'}{2} \partial_{\omega'}) \partial_{\omega'} \alpha[-\omega', \omega'] \} \end{aligned} \quad (26)$$

The point scatterer obeys a Langevin equation where the susceptibility and the force fluctuations are given by (25), and fluctuation-dissipation relations (9) and (11):

$$M_0 \ddot{q}(t) - \int_{-\infty}^\infty dt' \chi_T(t-t') q(t') = -Kq(t) + F(t) \quad (27)$$

The quasistatic responses (26) show that the quasistatic mass depends on the temperature, and that the mass entering the Langevin equation is the vacuum quasistatic mass M_0 :

$$M_T = M_0 + \frac{1}{2} \chi''_T[0] \quad (28)$$

Expanding around zero temperature, the first terms of the susceptibility χ_T and of the force commutator ξ_T can be obtained:

$$\begin{aligned}\chi_T[\omega] &= \chi_0[\omega] + \frac{i\pi T^2}{3\hbar} \omega \alpha[0, \omega] \\ \xi_T[\omega] &= \xi_0[\omega] + \frac{i\pi T^2}{6\hbar} \omega \gamma[0, \omega] \\ \xi_0[\omega] &= i\hbar \int_0^\omega \frac{d\omega'}{4\pi} \omega' (\omega - \omega') \gamma[\omega', \omega - \omega']\end{aligned}$$

Temperature corrections induce a damping force proportional to the velocity as in Brownian motion. The friction coefficient vanishes like T^2 near vacuum.

For temperature and frequencies well below the reflection cut-off ($T \ll \hbar\omega_c, \omega \ll \omega_c$), and if the scatterer can be considered as a perfect reflector over a large frequency interval, the limit of constant reflectivity ($\alpha[\omega, \omega'] = 2$) can be taken in (25), resulting in a simple form for the susceptibility:

$$\begin{aligned}\chi_T[\omega] &= i\xi_T[\omega] \\ \xi_T[\omega] &= \frac{\hbar}{6\pi} \omega^3 + \frac{2\pi T^2}{3\hbar} \omega\end{aligned} \quad (29)$$

In the classical limit ($\hbar\omega \ll T$), Brownian motion [16] is recovered with a diffusion coefficient for the particle's momentum (see (12)):

$$D = T \xi'_T[0] = \frac{2\pi T^3}{3\hbar} \quad (30)$$

In vacuum, the momentum diffusion vanishes (a consequence of momentum conservation) and a Langevin equation similar to the Abraham-Lorentz equation for a point charge follows:

$$M_0 \ddot{q} - \frac{\hbar}{6\pi} \ddot{\ddot{q}} = -Kq + F$$

A perfect reflector in vacuum is affected by the same instability problems as the point charge. Recalling (22), this illustrates the incompatibility between the infinite cut-off limit and the approximation neglecting recoil effects. In next section we show how stability and causality follow from a consistent treatment of radiative reaction.

C. Passivity

The causal nature of the force susceptibility results from that of the S-matrix [11], so that analytic properties can be used to recover the motional susceptibility from the mean force commutator and a dispersion relation. From eq.(25), the high frequency behavior of the susceptibility is dominated by the vacuum contribution. We shall assume in the following that the reflectivity is cutted off at high frequencies and that the vacuum susceptibility is such that $\chi_0[\omega]/\omega^3$ is a square integrable function. (A model of S-matrix satisfying causality, unitarity and transparency is for instance:

$$s = 1 + r \quad r[\omega] = -\frac{i\Omega}{\omega + i\Omega} \quad \frac{\chi_0[\omega]}{\omega^3} \sim -\frac{\hbar\Omega}{2\pi\omega} \quad for \quad \omega \gg \Omega$$

Then, the dispersion relation in the vacuum state can be written [19]:

$$\chi_0[\omega] = \omega^3 \int_{-\infty}^{\infty} \frac{d\omega'}{\pi\omega'^3} \frac{\xi_0[\omega']}{\omega' - \omega - i\epsilon}$$

so that at high frequencies:

$$\frac{\chi_0[\omega]}{\omega^3} \sim -\frac{\hbar\omega_c}{\omega} \quad \hbar\omega_c = \mu_0 = 2 \int_0^{\infty} \frac{dk}{\pi} \frac{\xi_0[k]}{k^3} < \infty \quad (31)$$

This defines the cut-off frequency ω_c introduced in (22). The thermal correction satisfies:

$$\begin{aligned} \chi_T[\omega] - \chi_0[\omega] &= \chi'_T[0]\omega + \omega^2 \int_{-\infty}^{\infty} \frac{d\omega'}{\pi\omega'^2} \frac{\xi_T[\omega'] - \xi_0[\omega'] - \xi'_T[0]\omega'}{\omega' - \omega - i\epsilon} \\ \frac{1}{2}\chi''_T[0] &= \mu_T - \mu_0 \\ \mu_T &= \int_{-\infty}^{\infty} \frac{dk}{\pi} \frac{\xi_T[k] - \xi'_T[0]k}{k^3} \end{aligned} \quad (32)$$

In a thermal state, the susceptibility (25) is recovered from the force commutator using the dispersion relation ($\xi''_T[0] = 0$):

$$\chi_T[\omega] = \chi'_T[0]\omega + \frac{1}{2}\chi''_T[0]\omega^2 + \omega^3 \int_{-\infty}^{\infty} \frac{d\omega'}{\pi\omega'^3} \frac{\xi_T[\omega'] - \xi'_T[0]\omega'}{\omega' - \omega - i\epsilon}$$

or in Laplace transforms:

$$\begin{aligned} -\frac{\chi_T\{p\}}{p} &= -i\chi'_T[0] + \frac{1}{2}\chi''_T[0]p - p^2 \int_{-\infty}^{\infty} \frac{dk}{\pi k^3} \frac{\xi_T[k] - \xi'_T[0]k}{p + ik} \\ &= \xi'_T[0] - \mu_0 p + \int_{-\infty}^{\infty} \frac{dk}{\pi} \frac{\xi_T[k]}{k(1+k^2)} \frac{1 + ipk}{p + ik} \end{aligned}$$

From (24) the coefficients entering the spectral decomposition can be seen to satisfy:

$$\xi'_T[0] \geq 0 \quad \frac{\xi_T[k]}{k} \geq 0$$

It results that for a point scatterer $-\chi/p$ is not a positive function (it has a negative residue for the pole at infinity).

Langevin equation for the scatterer (27) is solved in the frequency domain by (19), where the scatterer's impedance and admittance are given by:

$$\begin{aligned} Z\{p\} &= \xi'_T[0] + (M_0 - \mu_0)p + \frac{K}{p} \\ &+ \int_{-\infty}^{\infty} \frac{dk}{\pi} \frac{\xi_T[k]}{k(1+k^2)} \frac{1 + ipk}{p + ik} = Y\{p\}^{-1} \end{aligned}$$

The high frequency mass m of the scatterer is related to the quasistatic mass M_T through (see (28) and (32)):

$$m = M_T - \mu_T = M_0 - \mu_0$$

The scatterer's quasistatic mass is greater than its high frequency mass (see (31)). The difference is a mass induced by the field swept along the scatterer's motion, and vanishing at high frequencies where field and scatterer decouple (see the transparency condition). When (22) is satisfied, the high frequency mass is positive and the system's impedance has the spectral decomposition of a positive (or passive) function [20]. The admittance is also a positive function and the Langevin equation leads to stable and causal motions for a scatterer in a thermal state near vacuum [13]. Positivity of the high frequency mass, or a quasistatic mass greater than the induced mass, is the condition for stable and causal motions of the scatterer. It follows from that description that perfect reflection can only be consistent with an infinite quasistatic mass. For a finite mass scatterer, recoil effects must be taken into account before considering the reflection cut-off as infinite.

IV. CONCLUSION

Quantum Langevin equations possess general properties which are characteristic of linear response theory [5,14]. Coupling a particle to a field through radiation pressure leads to peculiar properties. The radiation pressure fluctuations and the radiative reaction force satisfy fluctuation-dissipation relations which exhibit a dependence of Langevin equation's kernel upon the input field state. The relations have been obtained here for thermal fields scattered by a point system. A similar situation occurs when irradiating a mirror with coherent light: the mirror satisfies a Langevin equation which depends on the intensity of the incident light. This has consequences on the ultimate sensitivity of interferometric measurements of positions [24].

Consistency of the simplified description in terms of reflection and transmission coefficients requires that the scatterer be transparent at frequencies greater than a reflection cut-off, corresponding to an energy smaller than the scatterer's mass. This inequality implies that the mass induced by field reaction is smaller than the quasistatic mass, i.e. that the high frequency mass of the scatterer is positive. This identifies with the condition for the Langevin equation to lead to stable and causal motions. This property can be considered as a consequence of the passivity of states near vacuum, i.e. of their incapacity to sustain 'runaway solutions' [13]. This must be compared with the case of a point charge, linearly coupled to the electromagnetic field. Renormalisation leads to an infinite negative bare mass, so that the Langevin equation possesses unstable solutions or violates causality [17]. In this case, recoil effects should explicitly be taken into account to get a consistent treatment of radiative reaction.

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